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ON THE NUMBER OF MULTIPLICATIONS ON SU (3) AND Sp (2)

BY
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Dedicated to Professor A. Komatu for his sixtieth birthday

1. **Introduction.** A multiplication on a space X with a base point $*$ is a map $\mu: X\times X\to X$ such that $\mu(*, x)=\mu(x, *)=x$ for any x of X. Two multiplications μ_1 and μ_2 are called homotopic if μ_1 is homotopic to μ_2 relative to $X\vee X$.

The enumeration problem of the homotopy classes of multiplications on a given H-space was studied by James [5] for spheres and by Arkowitz and Curjel [1] for finite CW-complexes. In fact, they showed that there exists a one-to-one correspondence between the homotopy set $[X\wedge X, X]$ and the set of homotopy classes of multiplications of a finite CW-complex X with a homotopy associative multiplication.

Moreover they proved that the following Lie groups have a finite number of nonhomotopic multiplications:

SO (n) for $n\leqq 16$ and $n\neq 10, 14$; SU (n) for $n\leqq 5$; Sp (n) for $n\leqq 7$; the representative of G_2, F_4 and E_7; and that all other classical groups and the representatives of the other exceptional structures have an infinite number of multiplications.

As is well known, SU (2) and Sp (1) have 12 distinct homotopy classes of multiplications, since these groups are homeomorphic to S^3 and since

$$[S^3 \wedge S^3, S^3] \cong \pi_6(S^3) \cong Z_{12}.$$

Recently Naylor [11] calculated the order of [SO (3) $\wedge$ SO (3), SO (3)] and showed that SO (3) has precisely 768 distinct homotopy classes of multiplications.

The purpose of this note is to compute the orders of [SU (3) $\wedge$ SU (3), SU (3)] and [Sp (2) $\wedge$ Sp (2), Sp (2)] and to prove

THEOREM 5.7. *There exist precisely* $2^{15}\cdot 3^9\cdot 5\cdot 7$ *distinct homotopy classes of multiplications on* SU (3).

THEOREM 6.6. *There exist precisely* $2^{20}\cdot 3\cdot 5^5\cdot 7$ *distinct homotopy classes of multiplications on* Sp (2).

Notation.

$[X, Y]$: the set of homotopy classes of continuous maps from X to Y.

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${}^{p}G$: the p-primary components of the abelian group G.

$\theta(G)$: the order of the group G.

$K^{(n)}$: the n-skeleton of a CW-complex K.

K/L: the quotient space of K by pinching L to a point for a CW-pair $K\supset L$.

$X \wedge Y$: the smash product $X\times Y/X \vee Y$.

$C_\alpha = Y \cup_\alpha CX$: the mapping cone of $\alpha \in [X, Y]$.

When we consider the mapping cone $Y\cup_f CX$ of a map $f\colon X\to Y$, we use often the homotopy class α of f in its place, since the homotopy type is determined by its homotopy class α.

The equality $X=Y$ for two spaces X and Y reads that X is homotopy equivalent to Y.

The paper is organized as follows. In §3 and §4, the cell structures of SU (3) $\wedge$ SU (3) and Sp (2) $\wedge$ Sp (2) are studied. Then the computations of the orders of [SU (3) $\wedge$ SU (3), SU (3)] and [Sp (2) $\wedge$ Sp (2), Sp (2)] are done in §5 and §6 respectively. The final section (§7) is devoted to the computations of ${}^{2}\pi_i(W_{5,2})$ for $i=15$, 16 and 17 which are needed in §5.

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2. **Preliminaries.** To begin with we recall some results on $\pi_{n+i}(S^n)$:

$\pi_n(S^n)\cong Z=\{\iota_n\}$ for $n\geqq 1$. $\pi_{n+1}(S^n)\cong Z_2=\{\eta_n\}$ for $n\geqq 3$.

$\pi_{n+2}(S^n)\cong Z_2=\{\eta_n^2\}$ for $n\geqq 3$. $\pi_{n+3}(S^n)\cong Z_{24}=\{\omega_n\}$ for $n\geqq 5$.

$\pi_{n+4}(S^n)=0$ for $n\geqq 6$. $\pi_{n+5}(S^n)=0$ for $n\geqq 7$.

$\pi_{n+6}(S^n)\cong Z_2=\{\nu_n^2\}$ for $n\geqq 5$, where $3\omega_n=\nu_n$.

$\pi_{13}(S^6)\cong Z_{60}=\{\Sigma''\}$. $\pi_{14}(S^7)\cong Z_{120}=\{\Sigma'\}$.

$\pi_{15}(S^8)\cong Z\oplus Z_{120}=\{\Sigma_8, S\Sigma'\}$.

$\pi_{n+7}(S^n)\cong Z_{240}=\{\Sigma_n\}$ for $n\geqq 9$,

where $S\Sigma''=2\Sigma'$, $S^2\Sigma'=2\Sigma_9$. Furthermore, we use the following notations: $15\Sigma''=\sigma''$, $15\Sigma'=\sigma'$ and $15\Sigma_9=\sigma_9$.

We also recall the well-known formulae

$$\pi_i(X\vee Y)\cong\pi_i(X)\oplus\pi_i(Y)\oplus\pi_{i+1}(X\times Y, X\vee Y). \tag{2.1}$$

$$[X\vee Y, Z]\cong[X,Z]\oplus[Y,Z]\quad\text{for a topological group } Z. \tag{2.2}$$

The groups SU (3) and Sp (2) are S^3-bundles over S^n where $n=5$ and 7 respectively. So they have the cell structure as follows [7]:

$$G = S^3\cup e^n\cup e^{n+3}.$$

It is also known that

(2.3) e^5 is attached to S^3 by η_3 in SU (3).

(2.4) e^7 is attached to S^3 by ω' in Sp (2), where ω' is the Blakers-Massey element and a generator of $\pi_6(S^3)\cong Z_{12}$. ($S^n\omega'=2\omega_{n+3}$ for $n\geqq 2$.)

According to James [6], in the suspended complex

$$SG = S^4\cup e^{n+1}\cup e^{n+4},$$

the cell e^{n+4} is attached by the element j_*x, where

$$j_*: \pi_{n+3}(S^4) \to \pi_{n+3}(S^4 \cup e^{n+1})$$

is the inclusion and $x \in \pi_{n+3}(S^4)$ is obtained from the characteristic element α for G by the Hopf construction. (This fact also follows from Lemma 2.32 of [12].)

Since the rotation group SO (4) splits as SO $(4) = S^3 \times$ SO (3), the characteristic elements are η_3 and ω' for $G =$ SU (3) and Sp (2) respectively. So the attaching maps of e^{n+4} in SG are $j_*(\nu_4\eta_7)$ and $j_*(2\nu_4^2)$ respectively. By Proposition 5.9 of [15], $S^2(\nu_4\eta_7) = 0$. Hence we obtain

$$S^3\,\mathrm{SU}\,(3) = S^6 \cup e^8 \vee S^{11}.$$

Similarly, $S^2\,\mathrm{Sp}\,(2) = S^5 \cup e^9 \vee S^{12}$ follows, since $S(2\nu_4^2) = 0$ by Proposition 5.11 of [15]. Thus, we have shown

LEMMA 2.1. (i) $S^n(\mathrm{SU}\,(3))$ *has the homotopy type of* $S^{n+3} \cup e^{n+5} \vee S^{n+8}$ *for* $n \geqq 3$.

(ii) $S^n(\mathrm{Sp}\,(2))$ *has the homotopy type of* $S^{n+3} \cup e^{n+7} \vee S^{n+10}$ *for* $n \geqq 2$.

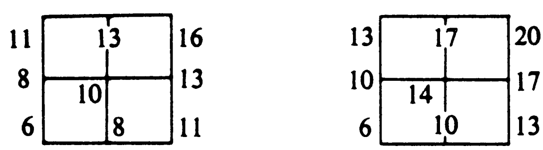
This lemma means that the cell e^8 in SU $(3)/S^3 = S^5 \cup e^8$ is not associated with S^5. So one has

(2.5) $$\mathrm{SU}\,(3)/S^3 = S^5 \vee S^8.$$

Similarly,

(2.6) $$\mathrm{Sp}\,(2)/S^3 = S^7 \vee S^{10}.$$

We will study the cell structure of $G \wedge G$, which evidently has 9 cells (see the diagram where the number indicates the dimension of the cell).



$K =$ SU (3) $\wedge$ SU (3) $M =$ Sp (2) $\wedge$ Sp (2)

For simplicity we set

$$K = \mathrm{SU}\,(3) \wedge \mathrm{SU}\,(3) \quad \text{and} \quad A = K^{(10)} = (S^3 \cup_{\eta_3} e^5) \wedge (S^3 \cup_{\eta_3} e^5).$$

$$M = \mathrm{Sp}\,(2) \wedge \mathrm{Sp}\,(2) \quad \text{and} \quad B = (S^3 \cup_{\omega'} e^7) \wedge (S^3 \cup_{\omega'} e^7).$$

Then we have

LEMMA 2.2. (i) $S^3(\mathrm{SU}\,(3) \wedge \mathrm{SU}\,(3)) = S^3A \vee (S^{14} \cup e^{16}) \vee (S^{14} \cup e^{16}) \vee S^{19}$.

(ii) $S^2(\mathrm{Sp}\,(2) \wedge \mathrm{Sp}\,(2)) = S^2B \vee (S^{15} \cup e^{19}) \vee (S^{15} \cup e^{19}) \vee S^{22}$.

Proof. By making use of Lemma 2.1, we have

$$
\begin{aligned}
S^3(\mathrm{SU}(3) \wedge \mathrm{SU}(3)) &= (S^3 \wedge \mathrm{SU}(3)) \wedge \mathrm{SU}(3) \\
&= (S^6 \cup e^8 \vee S^{11}) \wedge \mathrm{SU}(3) \\
\text{(i)} \qquad &= (S^6 \cup e^8) \wedge \mathrm{SU}(3) \vee (S^{11} \wedge \mathrm{SU}(3)) \\
&= (S^3 \cup e^5) \wedge (S^3 \wedge \mathrm{SU}(3)) \vee (S^{11} \wedge \mathrm{SU}(3)) \\
&= (S^3 \cup e^5) \wedge (S^6 \cup e^8 \vee S^{11}) \vee (S^{14} \cup e^{16} \vee S^{19}) \\
&= S^3A \vee (S^{14} \cup e^{16}) \vee (S^{14} \cup e^{16}) \vee S^{19}. \qquad \text{Q.E.D.}
\end{aligned}
$$

Similarly for (ii).

3. **The cell structure of** $\mathrm{SU}(3) \wedge \mathrm{SU}(3)$. First of all we study the structure of $A = K^{(10)}$.

Let C_η be the mapping cone of η_3 and let $1_c \colon C_\eta \to C_\eta$ be the identity map. Consider the map

$$1_c \wedge \eta_3 \colon C_\eta \wedge S^4 \to C_\eta \wedge S^3.$$

Clearly $A = C_\eta \wedge C_\eta$ is the mapping cone of $1_c \wedge \eta_3$. However, according to Lemma 3.5 of [15], (taking the trivial coextension, since $2\eta_6 = 0$), there exists an element η^* of $\pi_9(S^6)$ such that $1_c \wedge \eta_3 = (S^4 p)^*(i_* \eta^*)$ holds in $[S^4 C_\eta, S^3 C_\eta]$, where $i = S^6 \to S^3 C_\eta$ is the inclusion and $p \colon C_\eta \to S^5$ is the shrinking map. Hence we obtain

$$A = (S^6 \cup_{\eta_6} e^8) \cup_{1_c \wedge \eta_3} C(S^7 \cup_{\eta_7} e^9) = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\eta e^{10}$$

where $\eta = i_* \eta^* \vee \eta_8$.

We show that $\eta^* = n\nu_6$ with odd n. Clearly η^* belongs to ${}^2\pi_9(S^6) \cong Z_8 = \{\nu_6\}$. Consider the cohomology of A with Z_2-coefficient $H^*(A; Z_2)$. Let u and v be the generators of $H^3(C_\eta; Z_2)$ and $H^5(C_\eta; Z_2)$ respectively. Note that $S_q^2 u = v$ holds. Then the generators of $H^6(A; Z_2)$ and $H^{10}(A; Z_2)$ are represented by $u \wedge u$ and $v \wedge v$ respectively. By the Cartan formula we obtain

$$S_q^4(u \wedge u) = S_q^2 u \wedge S_q^2 u = v \wedge v.$$

Hence by Proposition 8.1 of [15] one deduces $\eta^* = n\nu_6$ with odd n. Thus

(3.1) $\quad A = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\eta e^{10}, \quad$ where $\eta = i_* n\nu_6 \vee \eta_8$ with $n =$ odd.

Recall that $K = A \cup e_1^{11} \cup e_1^{13} \cup e_2^{11} \cup e_2^{13} \cup e^{16}$. Consider the attaching element $\alpha_1 \in \pi_{11}(A)$. Then $S^3 \alpha_1$ is the attaching element of e_1^{14} in $S^3(K^{(11)})$, but it is trivial by (i) of Lemma 2.2. Hence $\alpha_1 = 0$. In fact, one has $S^3 \colon \pi_{10}(A) \cong \pi_{13}(S^3 A)$ since A is 5-connected. Similarly, the attaching element α_2 of e_2^{11} is trivial. Thus

$$K^{(11)} = A \vee S_1^{11} \vee S_2^{11}.$$

Next we study the attaching element β_1 of e_1^{13}:

$$\beta_1 \in \pi_{12}(K^{(11)}) \cong \pi_{12}(A) \oplus \pi_{12}(S_1^{11}) \oplus \pi_{12}(S_2^{11}).$$

For this we need

LEMMA 3.1. $S^3 \colon \pi_{12}(K^{(11)}) \to \pi_{15}(S^3(K^{(11)}))$ *is a monomorphism.*

Proof. It is sufficient to show that $S^3\colon \pi_{12}(A)\to\pi_{15}(S^3A)$ is a monomorphism. The following diagram is commutative and the two rows are exact.

$$\begin{array}{ccccccc} \pi_{13}(A, S^6) & \longrightarrow & \pi_{12}(S^6) & \longrightarrow & \pi_{12}(A) & \longrightarrow & \pi_{12}(A, S^6) \\ \downarrow S^3 & & \downarrow S^3 & & \downarrow S^3 & & \downarrow S^3 \\ \pi_{16}(S^3A, S^9) & \longrightarrow & \pi_{15}(S^9) & \longrightarrow & \pi_{15}(S^3A) & \longrightarrow & \pi_{15}(S^3A, S^9) \end{array} \tag{3.2}$$

Let $p\colon (A, S^6)\to(A/S^6, *)$ be a smashing map. Then we have a commutative diagram:

$$\begin{array}{ccc} \pi_{13}(A, S^6) & \xrightarrow{p_*} & \pi_{13}(A/S^6) \\ \downarrow S^3 & & \downarrow S^3 \\ \pi_{16}(S^3A, S^9) & \xrightarrow{(S^3p)_*} & \pi_{16}(S^3A/S^9) \end{array}$$

Since A/S^6 is 7-connected, we have that $S^3\colon \pi_{13}(A/S^6)\to\pi_{16}(S^3A/S^9)$ is an isomorphism. By the Blakers-Massey theorem (e.g. Corollary 3.3 of [10]), we obtain that p_* is epimorphic and that $(S^3p)_*$ is isomorphic. Hence the first S^3 in the diagram (3.2) is an epimorphism. Similarly one can show that the last S^3 in the diagram (3.2) is an isomorphism. The second S^3 is known ([15]) to be monomorphic. Thus by the four lemma (Lemma 5.6.9 of [3]) we obtain the lemma. Q.E.D.

The attaching element β_1 of e_1^{13} is of the form:

$$\beta_1 = \gamma \oplus x_1\eta_{11} \oplus x_2\eta_{11} \quad \text{with } \gamma\in\pi_{12}(A),\ x_i\in Z_2.$$

Then $S^3\beta = S^3\gamma \oplus x_1\eta_{14} \oplus x_2\eta_{14}$ is the attaching element of e_1^{16} in S^3K. We know, however, by (i) of Lemma 2.2 that e_1^{16} is attached not to S^3A but just to S_1^{14}. This implies that $S^3\gamma=0$, $x_1=1$ and $x_2=0$. Since S^3 is a monomorphism by Lemma 3.1, we obtain that $\beta_1=\eta_{11}$. Similarly the attaching element β_2 of e_2^{13} is checked to be η_{11}. Thus

$$K^{(13)} = A \vee (S_1^{11}\cup_{\eta_{11}} e_1^{13}) \vee (S_2^{11}\cup_{\eta_{11}} e_2^{13}).$$

The attaching element δ of e^{16} belongs to

$$\pi_{15}(K^{(13)}) \cong \pi_{15}(A) \oplus \pi_{15}(S_1^{11}\cup e_1^{13}) \oplus \pi_{15}(S_2^{11}\cup e_2^{13}).$$

On the other hand, by (2.5), we have that

$$\begin{aligned} &(\mathrm{SU}(3)/S^3) \wedge (\mathrm{SU}(3)) \\ &\qquad = (S^5\vee S^8)\wedge \mathrm{SU}(3) \\ &\qquad = S^5\,\mathrm{SU}(3) \vee S^8\,\mathrm{SU}(3) \\ &\qquad = (S_1^8\cup e_1^{10}) \vee S^{10} \vee (S_1^{10}\cup e_1^{13}) \vee S^{16} \quad \text{by Lemma 2.1.} \end{aligned} \tag{3.3}$$

This implies that δ has no components in $\pi_{15}(S_1^{11}\cup e_1^{13})$. Similarly it is checked that δ has no components in $\pi_{15}(S_2^{11}\cup e_2^{13})$, and hence

$$K = A\cup_\delta e^{16} \vee (S_1^{11}\cup e_1^{13}) \vee (S_2^{11}\cup e_2^{13}). \tag{3.4}$$

Then we will show that

$$A \cup_\delta e^{16}/S^6 = S_1^8 \times S_2^8 \cup e^{10}. \tag{3.5}$$

It is easily checked that the homomorphism induced by the inclusion:

$$\pi_{15}(S_1^8 \vee S_2^8) \to \pi_{15}((S_1^8 \vee S_2^8) \cup e^{10})$$

is an isomorphism, and hence $A/S^6 = (S_1^8 \vee S_2^8) \cup_\varepsilon C(S^9 \vee S^{15})$, where ε belongs to

$$[S^9 \vee S^{15}, S_1^8 \vee S_2^8] \cong \pi_9(S_1^8 \vee S_2^8) \oplus \pi_{15}(S_1^8) \oplus \pi_{15}(S_2^8) \oplus \pi_{16}(S_1^8 \times S_2^8, S_1^8 \vee S_2^8).$$

Thus the restriction of ε to S^{15} has the form

$$(\varepsilon | S^{15}) = \alpha_1 \oplus \alpha_2 \oplus b\tau,$$

where $\alpha_i \in \pi_{15}(S_i^8)$, $i=1, 2$, τ is a generator of $\pi_{16}(S_1^8 \times S_2^8, S_1^8 \vee S_2^8)$ and b is some integer.

To prove (3.5) it is sufficient to show $\alpha_i = 0$ and $b=1$. Let u_1 and u_2 be the generators of $H^8(A \cup e^{16}/S^6) \cong Z \oplus Z$, then the generator of $H^{16}(A \cup e^{16}/S^6)$ is a product $u_1 \wedge u_2$, since $H^{16}(K) \cong H^8(\mathrm{SU}(3)) \otimes H^8(\mathrm{SU}(3))$ holds. Suitably choosing orientations, we obtain $b=1$.

Consider the map $p: A^{(10)}/S^6 \xrightarrow{p_1} K/S^6 \xrightarrow{p_2} (\mathrm{SU}(3)/S^3) \wedge \mathrm{SU}(3)$, where p_1 is the inclusion and p_2 is the map shrinking $S_2^8 \vee S_2^{11}$ to a point in $K/S^6 = (S_1^8 \vee S_2^8) \cup C(S^9 \vee S^{15}) \vee (S_1^{11} \cup e_1^{13}) \vee (S_2^{11} \cup e_2^{13})$. Then we have that $p_*(\varepsilon | S^{15}) = \alpha_1$, which is the attaching element of e^{16} in $(\mathrm{SU}(3)/S^3) \wedge \mathrm{SU}(3)$. Hence $\alpha_1 = 0$ by (3.5). Similarly $\alpha_2 = 0$. Thus $(\varepsilon | S^{15})$ is a universal Whitehead product and hence (3.5) follows.

One can easily check that the cell e^{10} is attached to $S_1^8 \times S_2^8$ by (η_8, η_8) in $A \cup e^{16}/S^6$. Similarly $C(S_1^7 \vee S_2^7)$ is attached to S^6 by (η_6, η_6) in $A^{(8)}$.

Summing up these facts we state

PROPOSITION 3.2. *The smash product* $\mathrm{SU}(3) \wedge \mathrm{SU}(3)$ *has the cell structure of* $K = A \cup e^{16} \vee (S^{11} \cup_{\eta_{11}} e^{13}) \vee (S^{11} \cup_{\eta_{11}} e^{13})$, *where*

(i) $A = ((S^6 \cup_{\eta_6} e^8) \vee S^8) \cup_\alpha e^{10}$, $\alpha = i_* n\nu_6 \vee \eta_8$ *with odd* n,

(ii) $A \cup e^{16}/S^6 = S^8 \times S^8 \cup_\eta e^{10}$, $\eta = (\eta_8, \eta_8)$,

(iii) $A^{(8)} = S^6 \cup_{\eta_6} e^8 \vee S^8$.

Further we investigate the property of the suspended complex $S(A \cup e^{16})$ of $A \cup e^{16}$.

PROPOSITION 3.3. *Let* α *be the same as in* (i) *of Proposition* 3.2. *Then we have*

$$S(A \cup e^{16}) = SA^{(8)} \cup_\gamma C(S^{10} \vee S^{16}) \quad \textit{where } \gamma = S\alpha \vee \beta \circ \eta_{15}$$

with some β *of* $\pi_{15}(SA^{(8)})$.

The rest of this section is devoted to the proof of this proposition.

Let D be the complex $S(\mathrm{SU}(3)\wedge \mathrm{SU}(3))$. According to the above Proposition 3.2, the complex $S(A \cup e^{16})$ is a retract of D.

Let us recall (see §2) that $S(\mathrm{SU}(3))=S^4 \cup_{\xi'} C(S^5 \vee S^8)$ with $\xi'=\eta_4 \vee \nu_4\eta_7$.

Let 1_G be the identity map of SU (3). Then D has the homotopy type of

$$(3.6)\qquad \begin{aligned}(S^4 \cup_{\xi'} C(S^5 \vee S^8)) \wedge \mathrm{SU}(3) \\ = (S^4\,\mathrm{SU}(3)) \cup_{\xi} C(S^5\,\mathrm{SU}(3) \vee S^8\,\mathrm{SU}(3))\end{aligned}$$

with $\xi=1_G\wedge\eta_4\vee 1_G\wedge(\nu_4\eta_7)$.

By the naturality we have that

$$(3.7)\qquad 1_G \wedge (\nu_4\eta_7) = (1_G \wedge \nu_4)\circ(1_G \wedge \eta_7).$$

For simplicity we put $G_n=S^n\,\mathrm{SU}(3)$. Then we have $G_n=S^{n+3}\cup_{\eta_{n+3}} e^{n+5}\vee S^{n+8}$ for $n\geqq 3$ by Lemma 2.1.

Then $1_G\wedge\nu_4$ belongs to $[G_7, G_4]$ and $1_G\wedge\eta_7\in[G_8, G_7]$, where we have the isomorphisms by (2.1) and (2.2):

$$(3.8)\qquad [G_7, G_4] \cong [S^{10}\cup e^{12}, G_4] \oplus \pi_{15}(S^7\cup e^9) \oplus \pi_{15}(S^{12}).$$

$$(3.9)\qquad [G_8, G_7] \cong [S^{11}\cup e^{13}, G_7] \oplus \pi_{16}(S^{10}\cup e^{12}) \oplus \pi_{16}(S^{15}).$$

Let $j\colon S^7 \to S^7 \cup_{\eta_7} e^9$ and $k\colon S^{10}\to S^{10}\cup_{\eta_{10}} e^{12}$ be injections and consider the cofibering: $S^{10}\xrightarrow{k} S^{10}\cup_{\eta_{10}} e^{12}\to S^{12}$. Then we have the exact sequence

$$(3.10)\qquad \cdots\xleftarrow{\eta_{10}^*}\pi_{10}(G_4)\xleftarrow{k^*}[S^{10}\cup e^{12}, G_4]\longleftarrow\pi_{12}(G_4)\longleftarrow\cdots,$$

where $\pi_{10}(G_4)\cong\pi_{10}(S^7\cup e^9)$ by (2.1).

We need the following.

LEMMA 3.4. (i) $\pi_{10}(S^7\cup e^9)\cong Z_{12}$ *and generated by* $j_*\omega_7$,

(ii) $\pi_{16}(S^{10}\cup_{\eta_{10}} e^{12})\cong Z_2$ *generated by* $k_*\nu_{10}^2$,

(iii) k^* *is epimorphic.*

Proof. (i) and (ii) follow easily from the exact sequence of the pair $(S^7\cup e^9, S^7)$ and $(S^{10}\cup e^{12}, S^{10})$ respectively.

(iii) By definition we have

$$\begin{aligned}\eta_{10}^*(j_*\omega_7) &= j_*(\omega_7\eta_{10})\\ &= j_*(\nu_7\eta_{10})\\ &= 0 \quad \text{by (5.9) of [15].}\end{aligned}$$

So (iii) follows from the exactness of the sequence (3.10). Q.E.D.

Let δ be the attaching element of e^{16} in $A\cup e^{16}$ in Proposition 3.2. Since $S(A\cup e^{16})$ is a retract of $S(\mathrm{SU}(3)\wedge\mathrm{SU}(3))$ and since dimension of SA is 11, the attaching element $S\delta$ of e^{17} in $S(A\cup e^{16})$ is of the form $S\delta=\beta'\circ k_*\nu_{10}^2+\beta''\circ\eta_{15}$ where $\beta'\in[S^{10}\cup e^{17}, G_4]$ and $\beta''\in\pi_{15}(S^7\cup e^9)$.

By (i) and (iii) of Lemma 3.4 we have that

$$\beta' \circ k_* \nu_{10}^2 = (k^* \beta') \nu_{10}^2 = (x j_* \omega_7) \circ \nu_{10}^2 = x j_* \nu_7^3$$

with some integer x. Besides we have $\nu_7^3 = \bar{\nu}_7 \circ \eta_{15}$ by Lemma 6.3 of [15]. Namely

$$S\delta = (x j_* \bar{\nu}_7 + \beta'')\eta_{15}.$$

Putting $x j_* \bar{\nu}_7 + \beta'' = \beta \in \pi_{15}(SA^{(8)})$, we obtain the proposition. Q.E.D.

4. **The cell structure of** $\mathrm{Sp}(2) \wedge \mathrm{Sp}(2)$. Let n be a sufficiently large number. Let C_n' and C_n be the mapping cone of ω_n and $2\omega_n$ respectively. Let $1_n'$ and 1_n be the identity of C_n' and C_n respectively. For simplicity we set

$A_n = C_n' \wedge C_n'$, the mapping cone of $1_n' \wedge \omega_n$.

$B_n = C_n \wedge C_n$, the mapping cone of $1_n \wedge (2\omega_n)$.

Let j: $S^{2n} \to S^n C_n'$ and i: $S^{2n} \to S^n C_n$ be injections. Let $(2\omega_{2n+3})^{\sim} \in \pi_{2n+7}(S^n C_n')$ (or $(4\omega_{2n+3})^{\sim} \in \pi_{2n+7}(S^n C_n)$) be the coextension of $2\omega_{2n+3}$ (or $4\omega_{2n+3}$ respectively). Then by Lemma 3.5 of [15] we have that

$$A_n = (S^n C_n' \vee S^{2n+4}) \cup_{\alpha_n} e^{2n+8}, \tag{4.1}$$

where $\alpha_n = (j_* \alpha^* + (2\omega_{2n+3})^{\sim}) \vee \omega_{2n+4}$ with $\alpha^* \in \pi_{2n+7}(S^{2n})$.

$$B_n = (S^n C_n \vee S^{2n+4}) \cup_{\beta_n} e^{2n+8}, \tag{4.2}$$

where $\beta_n = (i_* \beta^* + (4\omega_{2n+3})^{\sim}) \vee 2\omega_{2n+4}$ with $\beta^* \in \pi_{2n+7}(S^{2n})$.

First we show that

(4.3) the restriction of α^* to ${}^2\pi_{2n+7}(S^{2n})$ is $x\sigma_{2n}$ with odd x.

Proof. By Proposition 8.1 of [15] we have

$$S_q^4 : H^n(C_n; Z_2) \cong H^{n+4}(C_n'; Z_2)$$

and hence by the Cartan formula we obtain

$$S_q^8 : H^{2n}(A_n; Z_2) \cong H^{2n+8}(B_n; Z_2).$$

Therefore, (4.1) follows by Proposition 8.1 of [15]. Q.E.D.

Next we show that

(4.4) the restriction of β^* to ${}^2\pi_{2n+7}(S^{2n})$ is $4x\sigma_{2n}$ with odd x.

Proof. We restrict ourself to the 2-primary components of the homotopy groups. It is easily checked that

$${}^2\pi_{2n+7}(S^n C_n' \vee S^{2n+4}) \cong Z_{16} \oplus Z_4 \oplus Z_8 = \{j_* \sigma_{2n}, (2\omega_{2n+3})^{\sim}, \nu_{2n+4}\}.$$

$${}^2\pi_{2n+7}(S^n C_n \vee S^{2n+4}) \cong Z_{16} \oplus Z_8 \oplus Z_8 = \{i_* \sigma_{2n}, (\omega_{2n+3})^{\sim}, \nu_{2n+4}\}.$$

Let F be a map: $B_n \to A_n$ such that

$F|S^{2n}$ is of mapping degree 1,

$F|S^{2n+4}$ and $F|e^{2n+4}$ are of mapping degree 2,

$F|e^{2n+8}$ is of mapping degree 4.

For the induced homomorphism

$$F_*: {}^2\pi_{2n+7}(S^nC_n \vee S^{2n+4}) \to {}^2\pi_{2n+7}(S^nC'_n \vee S^{2n+4})$$

we have that

(4.5) $$F_*(i_*\sigma_{2n}) = j_*\sigma_{2n}, \quad F_*(\omega_{2n+3})^\sim = (2\omega_{2n+3})^\sim, \quad F_*(\nu_{2n+4}) = 2\nu_{2n+4}.$$

Consider the following diagram:

$$\begin{array}{ccccc} {}^2\pi_{2n+7}(S^{2n+7}) & \xrightarrow[\beta_{n*}]{} & {}^2\pi_{2n+7}(S^nC_n \vee S^{2n+4}) & \longrightarrow & {}^2\pi_{2n+7}(S^nC_n \cup_{\beta_n} e^{2n+8}) \\ \downarrow \times 4 & & \downarrow F_* & & \downarrow \\ {}^2\pi_{2n+7}(S^{2n+7}) & \xrightarrow[\alpha_{n*}]{} & {}^2\pi_{2n+7}(S^nC'_n \vee S^{2n+4}) & \longrightarrow & {}^2\pi_{2n+7}(S^nC'_n \cup_{\alpha_n} e^{2n+8}) \end{array}$$

Let $y\sigma_{2n}$ be the restriction of β^* to ${}^2\pi_{2n+7}(S^{2n})$, where y is some integer. Then by the commutativity of the diagram we have that the 2-primary part of $F_*(\beta_{n*}\iota_{2n+7})$ is the 2-primary part of $\alpha_{n*}(4\iota_{2n+7})$. Namely,

$$j_*y\sigma_{2n} + 2\cdot(4\omega_{2n+3})^\sim + 2(2\nu_{2n+4}) = j_*4x\sigma_{2n} + 4\cdot(2\omega_{2n+3})^\sim + 4\nu_{2n+4}$$

and hence $j_*y\sigma_{2n} = j_*4x\sigma_{2n}$, since $2\cdot(4\omega_{2n+3})^\sim = 4\cdot(2\omega_{2n+3})^\sim$. One can easily see that $j_*: \pi_{2n+7}(S^{2n}) \to \pi_{2n+7}(S^nC'_n \vee S^{2n+4})$ is monomorphic. Therefore

$$y\sigma_{2n} = 4x\sigma_{2n}. \qquad \text{Q.E.D.}$$

Now we state the following proposition, which is a little weaker than the case SU (3) $\wedge$ SU (3).

PROPOSITION 4.1. *The smash product* Sp (2) $\wedge$ Sp (2) *has the homotopy type of* $M = B \cup (S^{13} \cup e^{17}) \cup (S^{13} \cup e^{17}) \cup e^{20}$, *where*

(i) $$B = (S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) \cup_\beta e^{14}, \quad \textit{with } \beta = i_*\sigma'' + (4\omega_9)^\sim \vee 2\omega_{10},$$

(ii) $$\begin{aligned} M/S^6 &= (S^{10} \times S^{10}) \cup_\gamma e^{14} \vee (S^{13} \cup_{2\omega_{13}} e^{17}) \vee (S^{13} \cup_{2\omega_{13}} e^{17}) \\ &= (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup_{2\omega_{13}} e^{17}) \vee (S^{13} \cup_{2\omega_{13}} e^{17}), \end{aligned}$$

where γ *is of type* $(2\omega_{10}, 2\omega_{10})$ *and* e^{20} *is attached to* $S^{10} \vee S^{10}$ *by a universal Whitehead product.*

(iii) $$M^{(10)} = B^{(10)} = S^6 \cup_{2\omega_6} e^{10} \vee S^{10}.$$

Proof. (i) First we prove (i). Let C be the mapping cone of ω' and 1_C the identity of C. Then $B = C \wedge C$ is a mapping cone of $1_C \wedge \omega': C \wedge S^6 \to C \wedge S^3$. For any coextension $(4\omega_9)^\sim$ of $4\omega_9$, by Lemma 3.5 of [15], there exists an element σ^* in $\pi_{13}(S^6)$ such that

$$1_C \wedge \omega' = (S^3p)^*(i_*\sigma^* + (4\omega_9)^\sim)$$

holds in $[C \wedge S^6, C \wedge S^3]$, where p is the shrinking map: $S^3 \cup e^6 \to S^6$ and i is the inclusion $S^6 \to C \wedge S^3 = S^6 \cup_{2\omega_6} e^{10}$. Therefore B has the homotopy type of

(4.6) $$B = (S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) \cup_\beta e^{14}$$

where $\beta = i_*\sigma^* + (4\omega_9)^\sim \vee 2\omega_{10}$ and $\sigma^* \in \pi_{13}(S^6) \cong Z_{60}$.

We will prove that

(4.7) $\sigma^* = x\sigma''$ with odd x, where σ'' is a generator of ${}^2\pi_{13}(S^6) \cong Z_4$.

Evidently $\sigma^* \in 5\pi_{13}(S^6) \cong Z_{12}$.

Recall that ${}^3\pi_{n+3}(S^n) \cong Z_3$ and ${}^3\pi_{n+7}(S^n) \cong Z_3$ are generated by $\alpha_1(n)$ and $\alpha_2(n)$ respectively for $n \geqq 3$ [15]. Let D be the mapping cone of $\alpha_1(6)$. Then it is easily checked that there exists an exact sequence:

$$0 \longrightarrow {}^3\pi_{13}(S^6) \xrightarrow{i_*} {}^3\pi_{13}(D) \xrightarrow{(S^4p)_*} {}^3\pi_{13}(S^{10}) \longrightarrow 0.$$

Since $\alpha_1(6) \circ \alpha_1(9) = 0$, there exists a coextension $[\alpha_1(9)]^{\sim}$ of $\alpha_1(9)$ in D. By definition $(S^4p)_*[\alpha_1(9)]^{\sim} = \alpha_1(10)$, which is a generator of ${}^3\pi_{13}(S^{10})$. Besides, by Proposition 1.8 of [15], the element $[\alpha_1(9)]^{\sim} \circ 3\iota_{13} = 3[\alpha_1(9)]^{\sim}$ belongs to $-i_*\{\alpha_1(6), \alpha_1(9), 3\iota_{12}\}$. This secondary composition contains an element $-i_*\alpha_2(6)$ by Proposition 4.17 of [14]. So we have

$$3[\alpha_1(9)]^{\sim} = -i_*\alpha_2(6) \mod -i_*\pi_{13}(S^6) \circ 3\iota_{13}.$$

Thus ${}^3\pi_{13}(D) \cong Z_9$ and it is generated by $[\alpha_1(9)]^{\sim}$. Therefore, in (4.6) we can choose the coextension $(4\omega_9)^{\sim}$ so that σ^* belongs to ${}^2\pi_{13}(S^6)$.

Since $S^{n-3}\omega' = 2\omega_n$ for sufficiently large n, we have $S^{2n-6}B = B_n$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
\pi_{14}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}, S^6) & \longrightarrow & \pi_{13}(S^6) & \xrightarrow{i_*} & \\
\downarrow S^{2n-6} & & \downarrow S^{2n-6} & & \\
\pi_{2n+8}(S^nC_n \vee S^{2n+4}, S^{2n}) & \longrightarrow & \pi_{2n+7}(S^{2n}) & \xrightarrow{i_*} &
\end{array}$$

$$\begin{array}{ccc}
\pi_{13}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}) & \longrightarrow & \pi_{13}(S^6 \cup_{2\omega_6} e^{10} \vee S^{10}, S^6) \\
\downarrow S^{2n-6} & & \downarrow S^{2n-6} \\
\pi_{2n+7}(S^nC_n \vee S^{2n+4}) & \longrightarrow & \pi_{2n+7}(S^nC_n \vee S^{2n+4}, S^{2n})
\end{array}$$

By making use of the Blakers-Massey theorem one can easily check that the first and the last S^{2n-6} are isomorphisms. Since the second S^{2n-6} is known to be monomorphic [15] we conclude by the four lemma that the third S^{2n-6} is monomorphic. Since $S^{2n-6}B = B_n$, we obtain that $S^{2n-6}(i_*\sigma^*)$ is $4x(i_*\sigma_{2n})$ by (4.4). On the other hand by Lemma 5.14 of [15], we have $4x(i_*\sigma_{2n}) = S^{2n-6}(i_*x\sigma'')$. As S^{2n-6} is monomorphic, we get that $i_*\sigma^* = x\sigma''$, and hence (4.7) follows.

(ii) We study the structure of $M/S^6 = (B/S^6) \cup (e_1^{13} \cup e_1^{17}) \cup (e_2^{13} \cup e_2^{17}) \cup e^{20}$. By virtue of (2.6) we obtain that

$$\begin{aligned}
(\mathrm{Sp}(2)/S^3) \wedge \mathrm{Sp}(2) &= (S^7 \vee S^{10}) \wedge \mathrm{Sp}(2) \\
&= S^7\,\mathrm{Sp}(2) \vee S^{10}\,\mathrm{Sp}(2) \\
&= (S^{10} \cup e^{14}) \vee S^{17} \vee (S^{13} \cup e^{17}) \vee S^{20} \quad \text{by Lemma 2.1.}
\end{aligned}$$

So it follows easily that

$$(M/S^6)^{(17)} = (B/S^6) \cup_\gamma C((S_1^{12} \cup e_1^{16}) \vee (S_2^{12} \cup e_2^{16})),$$

where $\gamma=(\gamma_1, \gamma_2)$ and $\gamma_i \in [S_i^{12} \cup e_i^{16}, B/S^6]$. On the other hand the double suspended complex $S^2((M/S^6)^{(17)})$ is the 19 skeleton of $(S^2M)/S^8$ and so by Lemma 2.2 it has the homotopy type of $(S^2B)/S^8 \vee (S_1^{15} \cup e_1^{19}) \vee (S_2^{15} \cup e_2^{19}) \vee S^{22}$. Therefore, $S^2\gamma_i=0$ in $[S^2(S_i^{12} \cup e_i^{16}), S^2(B/S^6)]$. The homomorphism

$$S^2\colon [S_i^{12} \cup e_i^{16}, B/S^6] \to [S^2(S_i^{12} \cup e_i^{16}), S^2(B/S^6)]$$

is an isomorphism, since B/S^6 is 9-connected. Thus γ_i is trivial and we obtain $(M/S^6)^{(17)}=(B/S^6) \vee (S_1^{13} \cup e_1^{17}) \vee (S_2^{13} \cup e_2^{17})$.

It is easily checked that e_i^{17} is attached to S_i^{13} by an element $2\omega_{13}$.

The same argument as in the proof of (ii) of Proposition 3.2 gives that the top cell e^{20} in M/S^6 is attached to $S_1^{10} \vee S_2^{10}$ by a universal Whitehead product. Thus $M/S^6=(S^{10} \times S^{10}) \cup e^{14} \vee (S^{13} \cup e^{17}) \vee (S^{13} \cup e^{17})$. It is easily seen that the attaching element of e^{14} is of type $(2\omega_{10}, 2\omega_{10})$.

One can restate the above fact as follows:

$$M/S^6 = (S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}) \vee (S^{13} \cup e^{17}) \vee (S^{13} \cup e^{17}).$$

(iii) The proof of (iii) is quite easy. Q.E.D.

For later use we state

COROLLARY 4.2. $SB=(S^7 \cup_{2\omega_7} e^{11} \vee S^{11}) \cup_{S\beta} e^{15}$, *where*

$$S\beta = i_*S\sigma'' + (4\omega_{10})^\sim \vee 2\omega_{11}.$$

5. **The order of** [SU (3) ∧ SU (3), SU (3)]. According to Arkowitz-Curjel (Lemma 2 of [1]) the order of the group [SU (3) ∧ SU (3), SU (3)] is equal to the number of homotopy classes of multiplications on SU (3). In this section we compute the order of this group.

By virtue of (2.2) and Proposition 3.2 we have

$$\text{(5.1)} \quad \begin{aligned}[\mathrm{SU}(3) \wedge \mathrm{SU}(3), \mathrm{SU}(3)] \cong [A \cup e^{16}, \mathrm{SU}(3)] &\oplus [S^{11} \cup_{\eta_{11}} e^{13}, \mathrm{SU}(3)] \\ &\oplus [S^{11} \cup_{\eta_{11}} e^{13}, \mathrm{SU}(3)].\end{aligned}$$

First we compute the order of $[S^{11} \cup_{\eta_{11}} e^{13}, \mathrm{SU}(3)]$.

Applying the functor [, SU (3)] to the cofibration sequence

$$S^{12} \xrightarrow[\eta_{11}]{} S^{11} \longrightarrow S^{11} \cup e^{13} \longrightarrow S^{13} \xrightarrow[\eta_{12}]{} S^{12} \longrightarrow \cdots,$$

we obtain the exact sequence:

$$\pi_{12}(\mathrm{SU}(3)) \xleftarrow[\eta_{11}^*]{} \pi_{11}(\mathrm{SU}(3)) \longleftarrow [S^{11} \cup e^{13}, \mathrm{SU}(3)] \longleftarrow \pi_{13}(\mathrm{SU}(3))$$
$$\xleftarrow[\eta_{12}^*]{} \pi_{12}(\mathrm{SU}(3)) \longleftarrow \cdots.$$

We quote from [9] that

$$\pi_{11}(\mathrm{SU}(3)) \cong Z_4([\nu_5^2]), \qquad \pi_{12}(\mathrm{SU}(3)) \cong Z_{60}([\sigma'''])$$,
$$\pi_{13}(\mathrm{SU}(3)) \cong Z_6(i_*\varepsilon')$$,

where ( ) denotes the generator of the 2-primary component and $[\alpha]$ is such an element that $p_*[\alpha]=\alpha$ for the projection $p\colon \mathrm{SU}(3) \to S^5$.

Then we have

LEMMA 5.1. *η_{11}^* and η_{12}^* are trivial.*

The proof will be given following Lemma 5.2.

As a corollary we have

PROPOSITION 5.2. $\theta([S^{11} \cup e^{13}, \mathrm{SU}(3)])=2^3\cdot 3$.

REMARK. One can show that

$$[S^{11} \cup e^{13}, \mathrm{SU}(3)] = \pi_{11}(\mathrm{SU}(3)) \oplus \pi_{13}(\mathrm{SU}(3))$$

but it is not necessary here.

Consider the complex Stiefel manifold $W_{5,2}=\mathrm{SU}(5)/\mathrm{SU}(3)$, which is known to be a S^7-bundle over S^9. And so $W_{5,2}=S^7 \cup_{\eta_7} e^9 \cup e^{16}$. We quote from §7 the following results.

PROPOSITION 7.1.

$$^2\pi_{15}(W_{5,2}) \cong Z_4 \oplus Z_2 = \{[\nu_9^2], i_*\bar{\nu}_7\}, \; 2[\nu_9^2] = i_*\varepsilon_7,$$
$$^2\pi_{16}(W_{5,2}) \cong Z_{16} = \{[2\sigma_9]\}, \; 8[2\sigma_9] = i_*\mu_7,$$
$$^2\pi_{17}(W_{5,2}) \cong Z_4 = \{i_*\nu_7\sigma_{10}\}.$$

In the above an element $[x] \in {}^2\pi_i(W_{5,2})$ is such that $p_*[x]=x \in \pi_i(S^9)$ for the projection $p\colon W_{5,2} \to S^9$.

Let $1\colon S^4\,\mathrm{SU}(3)=S^7 \cup_{\eta_7} e^9 \vee S^{12} \to W^{5,2}$ be a map such that $1|S^7 \cup e^9$ is an identity and $1|S^{12}$ is trivial. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & \pi_i(S^5) & & & & \\
 & & & \nearrow & p_* \| & \searrow & & & \\
\pi_i(S^3) & \xrightarrow{i_*} & \pi_i(\mathrm{SU}(3)) & \longrightarrow & \pi_i(\mathrm{SU}(3), S^3) & \longrightarrow & \pi_{i-1}(S^3) & \xrightarrow{i_*} & \pi_{i-1}(\mathrm{SU}(3)) \\
\downarrow S^4 & & \downarrow S^4 & & \downarrow S^4 & & \downarrow S^4 & & \downarrow S^4 \\
\pi_{i+4}(S^7) & \xrightarrow{i_*} & \pi_{i+4}(S^4\,\mathrm{SU}(3)) & \longrightarrow & \pi_{i+4}(S^4\,\mathrm{SU}(3), S^7) & \longrightarrow & \pi_{i+3}(S^7) & \xrightarrow{i_*} & \pi_{i+3}(S^4\,\mathrm{SU}(3)) \\
\| & & \downarrow 1_* & & \downarrow 1_* & & \| & & \downarrow 1_* \\
\pi_{i+4}(S^7) & \xrightarrow{i_*} & \pi_{i+4}(W_{5,2}) & \longrightarrow & \pi_{i+4}(W_{5,2}, S^7) & \longrightarrow & \pi_{i+3}(S^7) & \xrightarrow{i_*} & \pi_{i+3}(W_{5,2}) \\
 & & & \searrow & p_* \| & \nearrow & & & \\
 & & & & \pi_{i+4}(S^9) & & & &
\end{array}$$

For simplicity we denote by I the composition of the homomorphism

$$\pi_i(\mathrm{SU}(3)) \xrightarrow{S^4} \pi_{i+4}(S^4\,\mathrm{SU}(3)) \xrightarrow{1_*} \pi_{i+4}(W_{5,2}).$$

Then we have

LEMMA 5.2. (i) $I(i_*\varepsilon')=2i_*\nu_7\sigma_{10}$.
(ii) $I([\sigma'''])=4[2\sigma_9] \bmod i_*\mu_7$.
(iii) $I([\nu_5^2])=[\nu_9^2] \bmod i_*\bar{\nu}_7$.

Proof. We use the commutativity of the diagram:
(i) $I(i_*\varepsilon')=i_*(S^4\varepsilon')=i_*(2\nu_7\sigma_{10})$ by (7.10) of [15].
(ii) Since $p_*I([\sigma'''])=S^4p_*([\sigma'''])=8\sigma_9$, we have $I([\sigma'''])=4[2\sigma_9] \bmod i_*\mu_7$.
(iii) Similarly, the equality $p_*([\nu_5^2])=S^4p_*[\nu_5^2]=\nu_9^2$ implies that

$$I([\nu_5^2]) = [\nu_9^2] \bmod i_*\bar{\nu}_7.$$

Now we turn to the proof of Lemma 5.1.

Proof of Lemma 5.1. It is sufficient to show that $\eta_{11}^*[\nu_5^2]=\eta_{12}^*[\sigma''']=0$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\pi_{12}(\mathrm{SU}(3)) & \xleftarrow{\eta_{11}^*} & \pi_{11}(\mathrm{SU}(3)) & \longleftarrow & [S^{11}\cup e^{13}, \mathrm{SU}(3)] & \longleftarrow & \pi_{13}(\mathrm{SU}(3)) & \xleftarrow{\eta_{12}^*} & \pi_{12}(\mathrm{SU}(3)) \\
\downarrow I & & \downarrow I & & \downarrow I & & \downarrow I & & \downarrow I \\
\pi_{16}(W_{5,2}) & \xleftarrow{\eta_{15}^*} & \pi_{15}(W_{5,2}) & \longleftarrow & [S^{15}\cup e^{17}, W_{5,2}] & \longleftarrow & \pi_{17}(W_{5,2}) & \xleftarrow{\eta_{16}^*} & \pi_{16}(W_{5,2})
\end{array}$$

By Lemma 5.2, we have

$$I([\nu_5^2]\eta_{11}) = [\nu_9^2]\eta_{15} \bmod i_*\bar{\nu}_7\eta_{15},$$

where $i_*\bar{\nu}_7\eta_{15}$ is trivial by [9].

As the element $[\nu_9^2]$ one may choose a coextension $\tilde{\nu}_8^2$ of ν_8^2 in $S^7\cup e^9\subset W_{5,2}$, since $\eta_7\circ\nu_8^2=0$. Then by Proposition 1.8 of [15], $I([\nu_5^2]\eta_{11})=\tilde{\nu}_8^2\circ\eta_{15}$ belongs to $-i_*\{\eta_7, \nu_8^2, \eta_{14}\}$, which is easily proved to be zero. Thus $I([\nu_5^2]\eta_{14})=0$. By Lemma 5.2, the restriction of I on ${}^2\pi_{12}(\mathrm{SU}(3))$ is monomorphic. Therefore, we obtain $[\nu_5^2]\eta_{11}=0$.

We have $I([\sigma''']\eta_{12})=4[2\sigma_9]\eta_{16}=0 \bmod i_*\eta_7\mu_8$, where $i_*\eta_7\mu_8=0$ (see §7). Since I is monomorphic on ${}^2\pi_{13}(\mathrm{SU}(3))$ by Lemma 5.2 we have $[\sigma''']\eta_{12}=0$. Q.E.D.

Put $L=A\cup e^{16}/S^6=(A/S^6)\cup e^{16}$. By (2.2), we have

$$[S^9\vee S^{15}, S^8\vee S^8] \cong \pi_9(S^8\vee S^8)\oplus\pi_{15}(S^8\vee S^8).$$

Let h: $S^8\to S^8\vee S^8$ be the injection to the right S^8 and α be the universal Whitehead product in $\pi_{15}(S^8\vee S^8)$. Then by Proposition 3.2, we have

(5.2) $$L = (S^8\vee S^8)\cup_\gamma C(S^9\vee S^{15}) \quad \text{with } \gamma = h_*\eta_8\oplus\alpha.$$

Consider the cofibering sequence

(5.3) $$S^6\to A\cup e^{16}\to L\to S^7\to\cdots.$$

Applying the functor [, SU (3)] we get the exact sequence

$$(5.4)\quad \pi_6(\mathrm{SU}(3)) \xleftarrow{i^*} [A \cup e^{16}, \mathrm{SU}(3)] \longleftarrow [L, \mathrm{SU}(3)] \longleftarrow \pi_7(\mathrm{SU}(3)) \longleftarrow \cdots,$$

where $\pi_6(\mathrm{SU}(3)) \cong Z_6$ and $\pi_7(\mathrm{SU}(3))=0$ by [9].

So we will calculate the order of $[L, \mathrm{SU}(3)]$.

Consider the following cofibering sequence

$$(5.5)\quad S^9 \vee S^{15} \xrightarrow{\gamma} S^8 \vee S^8 \longrightarrow L \longrightarrow S^{10} \vee S^{16} \longrightarrow S^9 \vee S^9 \longrightarrow \cdots.$$

It follows the exact sequence

$$\pi_9(\mathrm{SU}(3)) \oplus \pi_{15}(\mathrm{SU}(3)) \xleftarrow[\gamma^*]{} [S^8 \vee S^8, \mathrm{SU}(3)] \longleftarrow [L, \mathrm{SU}(3)]$$

$$\longleftarrow \pi_{10}(\mathrm{SU}(3)) \oplus \pi_{16}(\mathrm{SU}(3)) \xleftarrow[(S\gamma)^*]{} [S^9 \vee S^9, \mathrm{SU}(3)] \longleftarrow \cdots.$$

Proposition 5.3. *γ^* and $(S\gamma)^*$ are trivial.*

Proof. Since $\gamma = h_*\eta_8 \oplus \alpha$ and $S\gamma = h_*\eta_9 \oplus S\alpha$, it is sufficient to prove

$$(5.6)\qquad \eta_8^*\pi_8(\mathrm{SU}(3)) = \eta_9^*\pi_9(\mathrm{SU}(3)) = 0,$$

$$(5.7)\qquad \alpha^*[S^8 \vee S^8, \mathrm{SU}(3)] = (S\alpha)^*[S^9 \vee S^9, \mathrm{SU}(3)] = 0.$$

By [9], $\pi_9(\mathrm{SU}(3)) \cong Z_3$ and hence η_8^* and η_9^* are trivial, since η_8 and η_9 are of order 2. (5.6) follows.

For any element x of $[S^8 \vee S^8, \mathrm{SU}(3)]$, the composition $x \circ \alpha = \alpha^*(x)$ is a Whitehead product in SU (3) and hence is trivial. Similarly $(S\alpha)^*=0$. (5.7) follows. Q.E.D.

It follows from Proposition 5.3 that

$$(5.8)\quad \theta([L, \mathrm{SU}(3)]) = \theta(\pi_8(\mathrm{SU}(3)))\cdot\theta(\pi_8(\mathrm{SU}(3)))\cdot\theta(\pi_{10}(\mathrm{SU}(3)))\cdot\theta(\pi_{16}(\mathrm{SU}(3))).$$

Recall [9] that $\pi_8(\mathrm{SU}(3)) \cong Z_{12}$, $\pi_{10}(\mathrm{SU}(3)) \cong Z_{30}$ and $\pi_{16}(\mathrm{SU}(3)) \cong Z_{252} \oplus Z_6$. We obtain

Corollary 5.4. $\theta([L, \mathrm{SU}(3)]) = 2^8\cdot 3^6\cdot 5\cdot 7$.

Finally we compute the order of $[A \cup e^{16}, \mathrm{SU}(3)]$. In the sequence (5.4) we have the commutativity:

$$\begin{array}{ccc} \pi_6(\mathrm{SU}(3)) & \xleftarrow{i^*} & [A \cup e^{16}, \mathrm{SU}(3)] \\ \| \wr & & \| \wr \\ \pi_7(B\,\mathrm{SU}(3)) & \xleftarrow{i^*} & [S(A \cup e^{16}), B\,\mathrm{SU}(3)] \end{array}$$

where i also means the inclusion $S^7 \to S(A \cup e^{16})$.

Lemma 5.5. *i^* is epimorphic.*

Proof. Let $j\colon S^4 \to B\,\mathrm{SU}(3)$ be the inclusion map. Then it is easily checked that the generator of $\pi_7(B\,\mathrm{SU}(3)) \cong \pi_6(\mathrm{SU}(3)) \cong Z_6$ is $j_*S\omega'$. The lemma is equivalent to the fact that $j_*S\omega'$ is extendable to $S(A \cup e^{16})$.

By Proposition 3.3 we have that

$$S(A \cup e^{16}) = SA^{(8)} \cup_\gamma C(S^{10} \vee S^{16}) \quad \text{where } \gamma = S\alpha \vee \beta \circ \eta_{15}.$$

Since $\pi_8(B\,\mathrm{SU}(3)) \cong \pi_7(\mathrm{SU}(3)) \cong 0$, $j_*S\omega'$ is extended to $SA^{(8)}$. Furthermore $j_*S\omega'$ is extended to $(S(A \cup e^{16}))^{(11)}$, since $j_*S\omega' \circ \alpha = j_*S\omega' \circ n\nu_7$ by (i) of Proposition 3.2 and since $j_*S\omega' \circ (n\nu_7) \in {}^2\pi_{10}(B\,\mathrm{SU}(3)) \cong {}^2\pi_9(\mathrm{SU}(3)) = 0$. The attaching element of e^{17} in $S(A \cup e^{16})$ is of the form $\beta \circ \eta_{15}$ by Proposition 3.3. Let ν be the extension of $j_*S\omega'$ to $(S(A \cup e^{16}))^{(11)}$. We must show that $\nu \circ \beta\eta_{15} = 0$. We need the following

(5.9) $$\eta_{15}^*\pi_{15}(B\,\mathrm{SU}(3)) = 0.$$

It is sufficient to show that

$$\eta_{14}^*({}^2\pi_{14}(\mathrm{SU}(3))) = 0.$$

In fact, ${}^2\pi_{14}(\mathrm{SU}(3)) \cong Z_4 \oplus Z_2 = \{[\nu_5^2] \circ \nu_{11}, i_*\mu'\}$ and hence $\eta_{14}^*({}^2\pi_{14}(\mathrm{SU}(3)))$ is generated by $[\nu_5^2] \circ \nu_{11} \cdot \eta_{14}$ and $i_*\mu'\eta_{14}$, where $[\nu_5^2]\nu_{11}\eta_{14} = 0$ by (5.9) of [15] and $i_*\mu'\eta_{14} = 0$ by Lemma 4.2 of [9].

The element $\nu\beta$ belongs to $\pi_{15}(B\,\mathrm{SU}(3))$ and hence $(\nu\beta)\eta_{15} = \eta_{15}^*(\nu\beta) = 0$ by (5.9). Q.E.D.

By (5.4) and Corollary 5.4 we have the following as a corollary of Lemma 5.5.

PROPOSITION 5.6. $\theta([A \cup e^{16}, \mathrm{SU}(3)]) = 2^9 \cdot 3^7 \cdot 5 \cdot 7$.

Therefore, by (5.1) and Proposition 5.2, we have obtained that

$$\theta([\mathrm{SU}(3) \wedge \mathrm{SU}(3), \mathrm{SU}(3)]) = 2^{15} \cdot 3^9 \cdot 5 \cdot 7.$$

Thus we have shown

THEOREM 5.7. *There exist precisely* $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ *distinct homotopy classes of multiplications on* SU (3).

6. **The order of** $[\mathrm{Sp}(2) \wedge \mathrm{Sp}(2), \mathrm{Sp}(2)]$. Consider the cofibering sequence ($M = \mathrm{Sp}(2) \wedge \mathrm{Sp}(2)$):

(6.1) $$S^6 \xrightarrow[j]{} M \longrightarrow M/S^6 \longrightarrow S^7 \longrightarrow SM \longrightarrow S(M/S^6) \longrightarrow S^8 \longrightarrow \cdots.$$

Applying the functor [, Sp (2)] we obtain the exact sequence:

(6.2) $$\begin{aligned}\pi_6(\mathrm{Sp}(2)) &\xleftarrow{j^*} [M, \mathrm{Sp}(2)] \longleftarrow [M/S^6, \mathrm{Sp}(2)] \longleftarrow \pi_7(\mathrm{Sp}(2))\\ &\xleftarrow{j^*} [SM, \mathrm{Sp}(2)] \longleftarrow [S(M/S^6), \mathrm{Sp}(2)] \longleftarrow \pi_8(\mathrm{Sp}(2)) \longleftarrow \cdots\end{aligned}$$

where $\pi_7(\mathrm{Sp}(2)) \cong Z$ and $\pi_6(\mathrm{Sp}(2)) \cong \pi_8(\mathrm{Sp}(2)) = 0$ ([9]).

By (2.2) and Proposition 4.1 we get

$$(6.3)\qquad \begin{aligned}[M/S^6, \mathrm{Sp}(2)] \cong {} & [(S^{10} \vee S^{10}) \cup C(S^{13} \vee S^{19}), \mathrm{Sp}(2)] \\ & \oplus [S^{13} \cup_{2\omega_{13}} e^{17}, \mathrm{Sp}(2)] \oplus [S^{13} \cup_{2\omega_{13}} e^{17}, \mathrm{Sp}(2)].\end{aligned}$$

First we compute the order of $[S^{13} \cup_{2\omega_{13}} e^{17}, \mathrm{Sp}(2)]$. Consider the sequence:

$$S^{16} \xrightarrow{2\omega_{13}} S^{13} \longrightarrow S^{13} \cup_{2\omega_{13}} e^{17} \longrightarrow S^{17} \xrightarrow{2\omega_{14}} S^{14} \longrightarrow.$$

Then we obtain the exact sequence:

$$\begin{aligned}\pi_{16}(\mathrm{Sp}(2)) \xleftarrow{(2\omega_{13})^*} \pi_{13}(\mathrm{Sp}(2)) \longleftarrow [S^{13} \cup e^{17}, \mathrm{Sp}(2)] \longleftarrow \pi_{17}(\mathrm{Sp}(2)) \\ \xleftarrow{(2\omega_{14})^*} \pi_{14}(\mathrm{Sp}(2)) \longleftarrow \cdots,\end{aligned}$$

where $\pi_{13}(\mathrm{Sp}(2)) \cong Z_4 \oplus Z_2 = \{[\nu_7]\nu_{10}, i_*\eta_3\mu_4\}$, $\pi_{14}(\mathrm{Sp}(2)) \cong Z_{1680}([2\sigma'])$,

$\pi_{16}(\mathrm{Sp}(2)) \cong Z_2 \oplus Z_2 = \{[\sigma'\eta_{14}]\eta_{15}, [\nu_7]\nu_{10}^2\}$ and $\pi_{17}(\mathrm{Sp}(2)) \cong Z_{40}([\nu_7]\sigma_{10})$.

Evidently $(2\omega_{13})^*$ is trivial, since $\pi_{16}(\mathrm{Sp}(2)) \cong Z_2 \oplus Z_2$. On the other hand we get $(2\omega_{14})^*[2\sigma'] = 4[\nu_7]\sigma_{10}$, since for the projection p: $\mathrm{Sp}(2) \to S^7$ we have

$$\begin{aligned}p_*([2\sigma'] \circ 2\omega_{14}) &= 4\sigma'\nu_{14} \\ &= 4\nu_7\sigma_{10} \quad \text{by (7.19) of [15]} \\ &= p_*(4[\nu_7]\sigma_{10})\end{aligned}$$

and since $p_*: {}^2\pi_{17}(\mathrm{Sp}(2)) \to {}^2\pi_{17}(S^7)$ is monomorphic by Lemma 5.2 of [9]. Thus the cokernel of $(2\omega_{14})^*$ is isomorphic to Z_{20}. Therefore we have obtained

PROPOSITION 6.1. $\theta([S^{13} \cup_{2\omega_{13}} e^{17}, \mathrm{Sp}(2)]) = 2^5 \cdot 5$.

For simplicity we put $N = (S^{10} \vee S^{10}) \cup_\xi C(S^{13} \vee S^{19})$, where

$$\begin{aligned}\xi = (2\omega_{10} \vee 4\omega_{10}) \oplus \delta \in \pi_{13}(S^{10} \vee S^{10}) \oplus \pi_{19}(S^{10} \vee S^{10}) \\ \cong [S^{13} \vee S^{19}, S^{10} \vee S^{10}]\end{aligned}$$

and δ is a universal Whitehead product by Proposition 4.1. We will compute the order of $[N, \mathrm{Sp}(2)]$.

We have the following cofibering sequence:

$$(6.4)\qquad S^{19} \vee S^{13} \xrightarrow[\xi]{} S^{10} \vee S^{10} \longrightarrow N \longrightarrow S^{20} \vee S^{14} \xrightarrow[S\xi]{} S^{11} \vee S^{11} \longrightarrow \cdots.$$

Therefore we obtain the exact sequence:

$$(6.5)\qquad \begin{aligned}\pi_{19}(\mathrm{Sp}(2)) \oplus \pi_{13}(\mathrm{Sp}(2)) \xleftarrow[\xi^*]{} [S^{10} \vee S^{10}, \mathrm{Sp}(2)] \longleftarrow [N, \mathrm{Sp}(2)] \\ \longleftarrow \pi_{20}(\mathrm{Sp}(2)) \oplus \pi_{14}(\mathrm{Sp}(2)) \\ \xleftarrow[(S\xi)^*]{} [S^{11} \vee S^{11}, \mathrm{Sp}(2)] \longleftarrow,\end{aligned}$$

where $\pi_{10}(\mathrm{Sp}(2)) \cong Z_{120}([\nu_7])$, $\pi_{11}(\mathrm{Sp}(2)) \cong Z_2 = \{i_*\varepsilon_3\}$, $\pi_{13}(\mathrm{Sp}(2)) \cong Z_4 \oplus Z_2 = \{[\nu_7]\nu_{10}, i_*\eta_3\mu_4\}$, $\pi_{14}(\mathrm{Sp}(2)) \cong Z_{1680}([2\sigma'])$, $\pi_{19}(\mathrm{Sp}(2)) \cong Z_2 \oplus Z_2 = \{i_*\mu_3\sigma_{12}, i_*\eta_3\bar{\varepsilon}_4\}$ and $\pi_{20}(\mathrm{Sp}(2)) \cong Z_2 \oplus Z_2 \oplus Z_2 = \{[\nu_7]\sigma_{10}\nu_{17}, i_*\eta_3\mu_4\sigma_{13}, i_*\bar{\mu}_3\}$ (see [9]).

Evidently $\xi^* = (2\omega_{10} \vee 4\omega_{10})^* \oplus \delta^*$ and $(S\xi)^* = (2\omega_{11} \vee 4\omega_{11})^* \oplus (S\delta)^*$.

PROPOSITION 6.2. (i) *$(S\xi)^*$ is trivial.*

(ii) *The kernel of ξ^* is isomorphic to $Z_{60} \oplus Z_{120}$.*

Proof. It is sufficient to prove

(6.6) δ^* and $(S\delta)^*$ are trivial.

(6.7) $(2\omega_{11} \vee 4\omega_{11})^*$ is trivial.

(6.8) The kernel of $(2\omega_{10} \vee 4\omega_{10})^*$ is isomorphic to $Z_{60} \oplus Z_{120}$.

The proof of (6.6) is same as in Proposition 5.3, since δ is a universal Whitehead product in $\pi_{19}(S^{10} \vee S^{10})$. (6.7) is clear since $\pi_{11}(\mathrm{Sp}(2)) \cong Z_2$. The image of $(2\omega_{10} \vee 4\omega_{10})^*$ is generated by $(2\omega_{10})^*[\nu_7] = 2[\nu_7]\nu_{10}$. Hence (6.8) follows. Q.E.D.

By (6.5) and Proposition 6.2 we obtain

PROPOSITION 6.3. $\theta([N, \mathrm{Sp}(2)]) = 2^{12} \cdot 3^3 \cdot 5^3 \cdot 7$.

Therefore by Proposition 6.1 we obtain

PROPOSITION 6.4. $\theta([M/S^6, \mathrm{Sp}(2)]) = 2^{22} \cdot 3^3 \cdot 5^5 \cdot 7$.

In order to enumerate the order of $[M, \mathrm{Sp}(2)]$ it is necessary to study the cokernel of the homomorphism in (6.2);

$$j^*: [SM, \mathrm{Sp}(2)] \to \pi_7(\mathrm{Sp}(2))$$

where $j: S^7 \to SM$ is the inclusion.

We have

PROPOSITION 6.5. *The cokernel of j^* is isomorphic to Z_{36}.*

It follows from this proposition and (6.2) that

$$0 \leftarrow [M, \mathrm{Sp}(2)] \leftarrow [M/S^6, \mathrm{Sp}(2)] \leftarrow Z_{36} \leftarrow 0$$

and hence

(6.9) $$\theta([M, \mathrm{Sp}(2)] = 2^{20} \cdot 3 \cdot 5^5 \cdot 7.$$

Namely by Lemma 2 of [1] we have shown

THEOREM 6.6. *There exist precisely $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ distinct homotopy classes of multiplications on* Sp (2).

The rest of this section is devoted to the proof of Proposition 6.5.

We have the following commutative diagram.

$$\begin{array}{ccc} \pi_7(\mathrm{Sp}(2)) & \xleftarrow{j^*} & [SM, \mathrm{Sp}(2)] \\ \| \wr & & \| \wr \\ \pi_8(B\,\mathrm{Sp}(2)) & \xleftarrow{j^*} & [S^2M, B\,\mathrm{Sp}(2)] \end{array}$$

Since $S^2M = S^2B \vee (S^{15} \cup e^{19}) \vee (S^{15} \cup e^{19}) \vee S^{22}$ by Proposition 4.1, we have

$$[S^2M, \mathrm{Sp}(2)] \cong [S^2B, \mathrm{Sp}(2)] \oplus [S^{15} \cup e^{19}, \mathrm{Sp}(2)] \oplus [S^{15} \cup e^{19}, \mathrm{Sp}(2)] \oplus \pi_{22}(\mathrm{Sp}(2)).$$

Let $j: S^7 \to SB = (S^7 \cup_{2\omega_7} e^{11} \vee S^{11}) \cup_{S\beta} e^{15}$ be the inclusion, where $S\beta = j_* S\sigma'' + (4\omega_{10})^{\sim} \vee 2\omega_{11}$. In order to prove Proposition 6.5 it is sufficient to show that

(6.7) the cokernel of $j^*: [SB, \mathrm{Sp}(2)] \to \pi_7(\mathrm{Sp}(2))$ is isomorphic to Z_{36}.

In other words, it is an extension problem of the generator $[12\iota_7]$ of $\pi_7(\mathrm{Sp}(2)) \cong Z$ to SB.

First we remark that we may take the coextension of $12\iota_7$ in $S^3 \cup_{\omega'} e^7 \subset \mathrm{Sp}(2)$ as an element $[12\iota_7]$.

We need the following

LEMMA 6.7. $i_*\{\omega', 12\iota_6, 2\omega_6\} = -i_*\alpha_2(3)$.

Proof. It is easily seen that the secondary composition $i_*\{\omega', 12\iota_6, 2\omega_6\}$ is a single element. Since $i_*: {}^2\pi_{10}(S^3) \to {}^2\pi_{10}(\mathrm{Sp}(2))$ is trivial, we have

$$i_*\{\omega', 12\iota_6, 2\omega_6\} = i_*\{\alpha_1(3), 12\iota_6, 2\alpha_1(6)\},$$

where $\{\alpha_1(3), 3\iota_6, \alpha_1(6)\} = \alpha_2(3)$ by Lemma 13.5 of [15]. Hence

$$i_*\{\omega', 12\iota_6, 2\omega_6\} = -i_*\alpha_2(3).$$

The composition $[12\iota_7] \circ 2\omega_7$ is contained in the secondary composition

$$-i_*\{\omega', 12\iota_6, 2\omega_6\}$$

by Proposition 1.8 of [15]. Hence by Lemma 6.7 we obtain

$$[12\iota_7] \circ 2\omega_7 = i_*\alpha_2(3). \tag{6.8}$$

Since $i_*\alpha_2(3)$ is of order 3, $3([12\iota_7]) = [12\iota_7] \circ 3\iota_7$ is extendable to the complex $S^7 \cup_{2\omega_7} e^{11} \vee S^{11}$. Let ξ be this extension. Consider the composition

$$\xi \circ S\beta = \xi \circ (j_* S\sigma'' + (4\omega_{10})^{\sim}).$$

LEMMA 6.8. (i) $\xi \circ j_* S\sigma'' = 4[2\sigma'] \bmod \{2i_*\mu' = 8[2\sigma']\}$.

(ii) $\xi \circ (4\omega_{10})^{\sim} = i_*\alpha_3(3) \bmod \{2i_*\mu' = 8[2\sigma']\}$.

Proof. (i) Since ξ is the extension of $[12\iota_7] \circ 3\iota_7$, we have that

$$\xi \circ j_* S\sigma'' = [12\iota_7] \circ 3\iota_7 \circ S\sigma''.$$

For the projection $p: \mathrm{Sp}(2) \to S^7$ we have $p_*([12\iota_7] \circ \sigma') = p_*(12\sigma')$ and hence $[12\iota_7] \circ \sigma' = 6[2\sigma'] \bmod \{i_*\mu'\}$. Thus we obtain

$$\xi \circ j_* S\sigma'' = 36[2\sigma'] = 4[2\sigma'] \quad \bmod \{2i_*\mu'\}.$$

(ii) By Proposition 1.7 of [15], the element $\xi \circ (4\omega_{10})^{\sim}$ is contained in the secondary composition $\{[12\iota_7] \circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2$. The indeterminacy of this is

$\pi_{11}(\mathrm{Sp}\,(2))\circ 4\omega_{11}\oplus[12\iota_7]\circ 3\iota_7\circ S^2\pi_{12}(S^5)$, which is easily checked to be generated by $8[2\sigma']=2i_*\mu'$. For any element x of $\{[12\iota_7]\circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2$, we have that $3x\in\{[12\iota_7], 2\nu_7, 4\nu_{10}\}_2\supset\{[12\iota_7], 2\nu_7, 4\iota_{10}\}_2\circ\nu_{11}$, which is a subset of $\pi_{11}(\mathrm{SU}\,(3))\circ\nu_{11}=0$. Hence

$$\{[12\iota_7]\circ 3\iota_7, 2\omega_7, 4\omega_{10}\}_2\subset {}^3\pi_{14}(\mathrm{Sp}\,(2)).$$

So we restrict ourself to the 3-primary components.

We have that

$$\{[12\iota_7]\circ 3\iota_7, 2\alpha_1(7), \alpha_1(10)\}_2\supset[12\iota_7]\circ\{3\iota_7, 2\alpha_1(7), \alpha_1(10)\}_2$$

by Proposition 1.8 of [15]

$$\ni[12\iota_7]\circ\alpha_2(7)$$

by Proposition 4.17 of [14].

On the other hand, by Proposition 1.7 of [15], we have

$$[12\iota_7]\circ\alpha_2(7)\equiv -i_*\{\alpha_1(3), 12\iota_7, \alpha_2(7)\}\qquad \mathrm{mod}\ 0.$$
$$= -i_*\alpha_3(3)\quad\text{by Lemma 13.5 of [15].}$$

Q.E.D.

Thus we get that

$$\xi\circ S\beta = 4[2\sigma']-i_*\alpha_3(3)\qquad \mathrm{mod}\ \{2i_*\mu' = 8[2\sigma']\},$$

which is of order 12. Hence 12ξ is extendable to SB. Namely just the 36 times of $[12\iota_7]$ is extendable to SB. Thus we have proved Proposition 6.5.

7. **Computation of some homotopy groups of $W_{5,2}$.** Let $W_{5,2}=\mathrm{SU}\,(5)/\mathrm{SU}\,(3)$ be the complex Stiefel manifold. For simplicity we set ${}^2\pi_{n+i}(S^n)=\pi^n_{n+i}$ and

$${}^2\pi_i(W_{5,2}) = W_i.$$

We will show the following.

PROPOSITION 7.1.

$${}^2\pi_{15}(W_{5,2})\cong Z_4\oplus Z_2=\{[\nu_9^2], i_*\bar\nu_7\},\quad \textit{where } i_*\varepsilon_7 = 2[\nu_9^2].$$
$${}^2\pi_{16}(W_{5,2})\cong Z_{16}=\{[2\sigma_9]\},\quad \textit{where } 8[2\sigma_9]=i_*\mu_7.$$
$${}^2\pi_{17}(W_{5,2})\cong Z_4=\{i_*\nu_7\sigma_{10}\}.$$

Proof. Associated with the bundle $S^7\xrightarrow{i}W_{5,2}\xrightarrow{p}S^9$ we have the exact sequence:

$$\pi^9_{18}\xrightarrow{\Delta}\pi^7_{17}\xrightarrow{i_*}W_{17}\xrightarrow{p_*}\pi^9_{17}\xrightarrow{\Delta}\pi^7_{16}\xrightarrow{i_*}W_{16}\xrightarrow{p_*}\pi^9_{16}\xrightarrow{\Delta}\pi^7_{15}$$
$$\longrightarrow W_{15}\longrightarrow\cdots,$$

where Δ is the boundary homomorphism.

Recall [15] that $\pi^9_{18}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{\sigma_9\eta^2_{16}, \nu^3_9, \mu_9, \eta_9\varepsilon_{10}\}$, $\pi^9_{17}\cong Z_2\oplus Z_2\oplus Z_2=\{\sigma_9\eta_{16}, \bar\nu_9, \varepsilon_9\}$, $\pi^9_{16}\cong Z_{16}=\{\sigma_9\}$, $\pi^9_{15}\cong Z_2=\{\nu^2_9\}$, $\pi^7_{17}\cong Z_8\oplus Z_2=\{\nu_7\sigma_{10}, \eta_7\mu_8\}$, $\pi^7_{16}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2=\{\sigma'\eta^2_{14}, \nu^3_7, \mu_7, \eta_7\varepsilon_8\}$, $\pi^7_{15}\cong Z_2\oplus Z_2\oplus Z_2=\{\sigma'\eta_{14}, \bar\nu_7, \varepsilon_7\}$ and $\pi^7_{14}\cong Z_8=\{\sigma'\}$.

We also recall the formula (see [9])

$$\Delta(S\alpha) = \eta_7\circ\alpha\quad\text{for any } \alpha\in\pi_i(S^8). \tag{7.1}$$

By (7.1) one can obtain immediately the following:

(7.2) $$0 \to Z_4 = \{\nu_7\sigma_{10}\} \to W_{17} \to 0,$$

(7.3) $$0 \to Z_2 = \{\mu_7\} \to W_{16} \to Z_8 = \{2\sigma_9\} \to 0,$$

(7.4) $$0 \to Z_2 \oplus Z_2 \to \{\bar{\nu}_7, \varepsilon_7\} \to W_{15} \to Z_2 = \{\nu_9^2\} \to 0.$$

Theorem 2.1 of [9] is useful in determining the extensions (7.3) and (7.4). In fact, the relation $i_*\{\eta_7, 2\sigma_8, 8\iota_{15}\} = i_*\mu_7$ implies $W_{16} \cong Z_{16} = \{[2\sigma_9]\}$ with $8[2\sigma_9] = i_*\mu_7$. The relation $i_*\{\eta_7, \nu_8^2, 2\iota_{14}\} = i_*\varepsilon_7$ means $W_{15} \cong Z_4 \oplus Z_2 = \{[\nu_9^2], i_*\bar{\nu}_7\}$ with

$$2[\nu_9^2] = i_*\varepsilon_7.$$ Q.E.D.

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